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The Ultra-Commutation Relations

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We discuss briefly the basic (integrable) representation of the ucr, comprising the operators A, its adjoint A^+, and N (which is equal to N^+), satisfying $AN - NA = A$. There are no additional relations between the operators, in general. The ucr include the ccr, car, deformed bosons and fermions, and many other systems as special cases. The principal structure theorem asserts that every integrable representation of the ucr is determined by a sequence generalizing the $[n]$-sequence of deformation theory.

1. INTRODUCTION

A great many "quantum" systems are now in fashion—strings, quantum groups, deformed oscillators, and so on—so that commutation relations other than those for bosons or fermions are now of considerable interest. But it must be noted that in the "deformed" industry in particular, the "independent rediscovery" rate in publications is significant, as is the reappearance of known results in "unitary disguise."

With this in mind, we present a way of classifying many of the deformed systems which provides a sure means of distinguishing one from another, and has the additional merits of being easy to understand and easy to apply—although not easy to prove!

The key to the proposal is is to loosen—but not break completely—the connection between the number operator and the raising and lowering operators. We no longer require that $N = A^+A$, but retain $AN - NA = A$. We call this latter relation (and its adjoint) the ultra commutation relations, or ucr.

Analysis of the representations of these relations involves the usual technical complications involving families of unbounded operators. So to

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find and classify the basic representations requires a certain mathematical care. Proofs, details, and further discussion will be found in a forthcoming article (Dubin *et al.*, 1997).

2. THE UCR ALGEBRA AND REPRESENTATIONS

We start from the underlying abstract algebra leading to the ucr:

Definition. The ucr polynomial algebra for one degree of freedom is the unital complex *-algebra, denoted $\mathcal{A}$, of all polynomials in two indeterminates α and ν with $\nu = \nu^+$ and subject to the relation $\alpha\nu - \nu\alpha = \alpha$, which we call the ultra-commutation relation (abbreviated to ucr).

Proposition. The algebra elements $\sigma = \alpha^+\alpha$ and $\tau = \alpha\alpha^+$ commute with ν.

By $\mathcal{D}$ we always mean a dense subspace of a Hilbert space $\mathcal{H}$, and by L^+ we mean the set of all operators B: $\mathcal{D} \to \mathcal{D}$ for which B^*: $\mathcal{D} \to \mathcal{D}$. A *-subalgebra $\mathcal{B}$ of L^+ determines a (graph) topology generated by the seminorms $\|f\|_b = \|bf\|$, $f \in \mathcal{D}$, $b \in \mathcal{B}$. By a *-representation $(\pi, \mathcal{D})$ of $\mathcal{A}$ we mean an algebra homomorphism π: $\mathcal{A} \to L^+(\mathcal{D})$, where, if we write $\pi(x)^+$ to mean $\pi(x)^*$ restricted to $\mathcal{D}$, then $\pi(x^+) = \pi(x)^+$. When it is clear which representation is being considered we use the notation $A = \pi(\alpha)$, $A^+ = \pi(\alpha^+)$, and

$$S = \pi(\sigma) = A^+A, \qquad T = \pi(\tau) = AA^+, \qquad N = \pi(\nu).$$

As *-representations preserve relations among generators, the ucr hold on $\mathcal{D}$ in the sense that

$$ANf - NAf = Af, \qquad A^+Nf - NA^+f = -A^+f, \qquad f \in \mathcal{D}.$$

The *-symmetry of σ, τ, and ν is preserved, so

$$S = S^+ \qquad T = T^+, \qquad N = N^+,$$

but it does not follow that S, T, N are essentially self-adjoint; nor if they are, do they necessarily commute strongly, i.e., their spectral projections mutually commute.

We want to consider N as a generalized number operator, so we shall consider only those representations in which $\bar{S}$, $\bar{T}$, and $\bar{N}$ are essentially self-adjoint and commute strongly, and $\bar{N}$ has a spectrum consisting of isolated eigenvalues of finite multiplicity. A further pathology is excluded by demanding that the domain $\mathcal{D}$ be stable under S, T, and N. These requirements are not vacuous, as we can construct representations of the ucr in which they do not hold.

The spectrum of $\bar{N}$ will not be a subset of the positive integers in general. We have shown that from every representation of the above sort we can extract a family of subrepresentations $\{\pi_\mu, \mathcal{D}_\mu$: $\mu \in [0, 1)\}$ such that the spectrum of $\overline{N} - \mu I_\mu$ is a subset of N ∪ {0}, all but a countable subfamily of these representations (at most) are trivial, and $(\pi, \mathcal{D})$ is a direct sum of the $(\pi_\mu, \mathcal{D}_\mu)$. Hence by considering ucr representations $(\mathcal{D}_\mu, A_\mu^+, S_\mu, T_\mu, N_\mu - \mu I)$, we are assured that the number operator is of the desired type. We call this a shift-reduction procedure.

There remains one last type of restriction to consider for the representations. The number operator defines eigenspaces through its spectral decomposition, but their dimensions are unrelated to one another. It is best, therefore, to begin the analysis by restricting them to be one dimensional, and come to the general case by direct sums. We call these representations integrable.

An r-dimensional integrable representation, then, is a *-representation of $\mathcal{A}$ on an r-dimensional Hilbert space $\mathcal{H}$, with $N = \Sigma_{n=0}^{r-1} nP_n$ and $I = \Sigma_{n=0}^{r-1} P_n$, where dim $P_n = 1$ for all $0 \le n \le r - 1$. For each $n \ge 0$, choose a unit vector Ω_n in the image of P_n, so the set $\{\Omega_n\colon 0 \le n \le r - 1\}$ is an orthonormal basis for $\mathcal{H}$. Using this basis, there is an obvious isomorphism between $\mathcal{H}$ and $\mathcal{C}^r$ under which $\pi(\mathcal{A})$ is isomorphic to a *-subalgebra of $M_r(\mathcal{C})$. It turns out that the general form of the matrices representing $\pi(\mathcal{A})$ in this way can be written down quite explicitly.

When $\mathcal{H}$ is infinite dimensional, there is still a choice of common domain $\mathcal{D}$ to be made. We choose the largest reasonable one, in the following sense. First choose eigenvectors Ω_n of N as for r dimensions, and now we have a countably infinite basis for $\mathcal{H}$. Start with the smallest domain, the linear span of these eigenvectors $\mathcal{D}_0$, and consider representations π_0 on this space. (Any larger representation of this type can be restricted to $\mathcal{D}_0$.) The representations we want are the adjoints of these minimal representations. This is a technical requirement of some delicacy, and we have shown (Dubin *et al.*, n.d.) that the result is a *-representation of the best sort (countably dominated and strongly self-adjoint, in the language of op-* algebras).

We consider only integrable representations from now on.

3. STRUCTURE THEOREMS

Because we have such a simple spectral structure for the number operator, we can combine it with the ucr to determine the action of the operators A and A^+. Our technical conditions in the infinite-dimensional case assure us that the calculations are mathematically well defined.

The result is that for any integrable representation, there exists a sequence $(\mu_n)_{n\ge0}$ of complex numbers such that $A^+\Omega_n = \mu_{n+1}\, \Omega_{n+1}$, $n \ge 0$, and $A\Omega_n = \mu_n\, \Omega_{n-1}$, $n \ge 1$, with $A\Omega_0 = 0$, which we call the ladder operations. We refer to A as a lowering operator and A^+ as a raising operator.

It then follows that $\overline{S} = \Sigma_{n=1}^{\infty} |\mu_n|^2 P_n$, $\overline{T} = \Sigma_{n=0}^{\infty} |\mu_{n+1}|^2 P_n$. A consistent convention we adopt is to suppose in all cases that $\mu_0 = 0$. Otherwise, there is no requirement that the constants μ_n be nonzero.

Recall that we came to the spectrum of N by shifting by a constant. Shifting N back by μ, we still have $A\Omega_0 = 0$, but now $(N + \mu I)\Omega_0 = \mu\Omega_0$ does not vanish.

Regarding the analytic structure of any integrable representation (π, D), we have already noted that it is self-adjoint, so $\pi(\alpha^+)^* = \pi(\alpha)$, $\pi(\alpha)^* = \pi(\alpha^+)$. The graph topology on $\mathcal{D}$ determined by $\pi(\mathcal{A})$ is Fréchet.

We can characterize $\mathcal{D}$ explicitly, if messily, as follows: the vector $\Sigma_{n=0}^{\infty} c_n\Omega_n$ belongs to $\mathcal{D}$ if and only if (c_n) belongs to the sequence space

$$\Xi = \{x \in l^2 \colon u_{y_1, y_2, \ldots, y_m}(x) < \infty \; y_1, y_2, \ldots, y_m \in \mathcal{C}\}$$

where

$$\mathcal{C} = \{x\nu^r \colon x \text{ is a monomial in } \alpha, \alpha^+; \; r = 0, 1, 2, \ldots\},$$

$$u_{y_1, y_2, \ldots, y_m}(x)^2 = \sum_{n=0}^{\infty} ([1 + K_n(y_1) + K_n(y_2) + \cdots + K_n(y_m)]^2)|x_n|^2)$$

and

$$K_n(y) = \|\pi(y)\Omega_n\|^2$$

are seminorms. With this topology on Ξ, it is isomorphic to $\mathcal{D}$ as a locally convex space. (This is all trivial for finite-dimensional representations.)

In what follows, it will prove useful and illuminating to to introduce the notation $[n] = |\mu_n|^2$, $n \geq 0$, for an integrable representation, where the symbol $[n]$ is to be read as “box-n.” This extends to notation used in the q-calculus, originally due to Jackson.

The critical structure theorem identifies the unitary equivalence classes of integrable representations in a simple fashion:

The Unitary Equivalence Theorem. Two integrable (resp. minimal) representations are unitarily equivalent if and only if their $[n]$ sequences coincide.

We have not mentioned irreducibility yet, and it must not be thought that integrable representations are necessarily irreducible. For integrable representations the many possible complications which can occur in op-*algebras are not present: the principal bounded commutants all coincide, and so we may write, simply, $\pi(\mathcal{A})'$ for the resulting W^*-algebra. Hence (π, D) is irreducible (in any sense) if and only if the only reducing subspaces of $\mathcal{D}$ are $\{0\}$ and $\mathcal{D}$ itself; or, there is no nontrivial decomposition of π as a direct sum of integrable particle *-representations of the ucr; or $\pi(\mathcal{A})'$ consists of scalars.

But this does not tell us how to recognize when a ucr representation is irreducible. For that, we have shown the following: an integrable representation is irreducible if and only if $[n] \neq 0$ for all $n \in \mathbb{N}$.

We may now state the most important practicable results of the mathematical theory: given an integrable representation of the ucr, in however unusual a form, simply by calculating the normalization constants $[n]$ we can immediately learn what type of system is being described and read off the irreducible subrepresentations.

4. EXAMPLES OF INTEGRABLE REPRESENTATIONS

4.1. The CCR

There is only one integrable representation of the ccr, namely the Schrödinger representation. It is irreducible and the common domain is $\mathcal{S}(\mathbb{R})$, the eigenfunctions Ω_n being the Hermite–Gaussian functions. As $\mu_n = n^{1/2}$, the box sequence is $[n] = n$ for all $n \geq 0$.

4.2. Bounded Operators

Suppose the sequence $(\mu_n)_{n\geq 0}$ is bounded: $\sup_{n\geq 0}[n] = \beta < \infty$. Then N is not bounded, but the other operators are:

$$\|A\| \leq \beta^{1/2}, \qquad \|A^+\| \leq \beta^{1/2}, \qquad \|S\| \leq \beta, \qquad \|T\| \leq \beta$$

We can choose the vectors Ω_n to be the Hermite–Gaussian functions, so that $\mathcal{D} = \mathcal{S}(\mathbb{R})$, but A and A^+ are no longer those found in the ccr.

As an illustration, suppose $\mu_n = 1$ for all $n \geq 1$. Then $AA^+ - A^+A = P_0$, the projection onto Ω_0. A realization of this is on Hardy space H^2 over the unit circle, with $\Omega_n(e^{i\theta}) = e^{in\theta}$, raising and lowering operators

$$(A^+g)(e^{i\theta}) = e^{i\theta}g(e^{i\theta}), \qquad (Ag)(e^{i\theta}) = P^+e^{-i\theta}g(e^{i\theta})$$

and $S = P^+ - P_0$, $T = P^+$. Here P^+ is the Szegö projection.

4.3. Deformed Maths Bosons and Fermions

These are integrable representations satisfying the commutation relation

$$AA^+ - qA^+A = I, \qquad |q| < 1$$

in addition to the ucr. Depending on the value of q, different sorts of systems are described in this way. Negative q describes q-deformed maths fermions, and positive q describes q-deformed maths bosons. The term ‘maths’ distinguishes these systems from their q-deformed ‘physics’ relatives described below.

The box sequences for these systems involves the geometric series polynomial functions

$$p_n(x) = 1 + x + x^2 + \cdots + x^{n-1} = \frac{1 - x^n}{1 - x}, \qquad x \in \mathscr{C}\setminus\{1\}$$

For given q the $[n]$ sequence is $[0] = 0$ and

$$[n] = p_n(q) = \frac{1 - q^n}{1 - q}, \qquad n \geq 1$$

Clearly, these representations are irreducible. They are bounded, with

$$\beta = \begin{cases} 1, & -1 < q < 0 \quad \text{(maths fermions)} \\ (1 - q)^{-1} & 0 \leq q < 1 \quad \text{(maths bosons)} \end{cases}$$

Hence we can choose the vectors Ω_n to be the Hermite–Gaussian functions and $\mathfrak{D} = \mathscr{S}(\mathbb{R})$. The case $q = 0$ is the example $[n] = 1$ discussed above. The case $q = -1$ yields $[2] = 0$, which is the car, which we discuss below. The case $q = 1$ yields $[n] = n$ for all $n \in \mathcal{N}$, which is the ccr. Thus, this family interpolates between the ccr and car through bounded representations.

A variant type of system is obtained by using the commutation relation

$$AA^+ - qA^+A = cI, \qquad c > 0$$

This yields $[n] = cp_n(q)$ for all $n \geq 1$ and $|q| < 1$. These are quantum hyperboloids, with noncommuting coordinates A, A^+.

The twisted ccr for one degree of freedom is a q-maths system. Their irreducible unitary representations have been found by Pusz and Woronowicz for $q > 0$ and by Schmüdgen for $|q| = 1$. (They analyzed such systems with more than one degree of freedom as well, a rather more difficult problem.)

4.4. Deformed Physics Bosons and Fermions

These are integrable representations satisfying the commutation relations $AA^+ - qA^+A = q^{-N}$, $|q| < 1$, in addition to the ucr. Negative q are q-deformed physics fermions, and positive q are q-deformed physics bosons. For given q, the $[n]$ sequence is $[0] = 0$ and

$$[n] = \frac{1}{q^{n-1}} p_n(q^2) = \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad n \geq 1$$

From this we see that, in fact, there are no q-deformed physics fermions, although q-deformed physics bosons exist. For such systems, $\mathfrak{D}$ is infinite dimensional, the representation is irreducible, with A and A^+ unbounded, since $[n] \to \infty$ as $n \to \infty$ for any $0 < q < 1$.

5. CONCLUSION

We may give explicit realizations as matrices in the case of finite r-dimensional representations. For example, when $r = 2$ we recover only the representation of the car discovered by Jordan and Wigner in 1928.

This completes our discussion here. In Dubin *et al.* (n.d.) the reader will find a discussion of spatial and kernel realizations of integrable representations by differential operators as well as proofs and details of the analysis.

REFERENCE

Dubin, D. A., Hennings, M. A., and Solomon, A. I. (1997). Integral representations of the ultra-commutation relations, *J. Math. Phys.* **38**, 3238–3262.